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Inhomogeneous Cosmologies: The Cosmic Peeling-Off Property of Gravity

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Asymptotic expansions are used to study the late-time behavior of the Riemann tensor for a large class of inhomogeneous cosmological models. These models include the unidirectional inhomogeneous generalizations of Bianchi type I-VII universes. The cosmic analog of the peeling-off theorem, familiar from general relativity, is formulated. It is also shown that chaotic behavior near the initial singularity transforms itself into pure gravitational radiation at late times.

1. INTRODUCTION

In the last few years, the possibility of the existence of gravitational-radiation background in the Universe has been extensively discussed by many authors (Rossi and Zimmerman, 1976; Carr, 1980; Zimmerman and Hellings, 1980; Zeldovich and Novikov, 1983; Adams et al., 1982; Carr and Verdaguer, 1983). From the theoretical point of view such discussions became possible owing to the considerable progress in the study of inhomogeneous and anisotropic cosmologies. There is no evidence that our Universe, which is reasonably described at this stage of evolution by one of the Friedmann homogeneous and isotropic models, has passed a regular expansion during its early ages. On the contrary, it looks aesthetically more appealing to consider initially irregular expansion, which during the process of evolution has been smoothed by various damping processes.

One of the theoretically predicted consequences of primordial irregularities is the emergence of a cosmological gravitational radiation background. Such a behavior has been studied by Adams et al. (1982) and by Carr and Verdaguer (1983) using two different techniques. Adams et al. studied solutions describing gravitational waves on homogeneous Bianchi backgrounds, whereas Carr and Verdaguer used the so-called "soliton"

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technique, developed by Belinskii and Zakharov (1979), to study the behavior of inhomogeneous cosmological models. In both approaches it was possible to find exact solutions with initial chaotic behavior near the singularity which transforms into gravitational radiation at late times.

In this paper we present the cosmic analog (Carmeli and Feinstein, 1984; Carmeli and Charach, 1980) of the familiar peeling-off property of the classical gravitational field (Sachs, 1962). This will enable us to find the late-time behavior of a large class of inhomogeneous cosmological universes. One of our results will be the confirmation of the prediction mentioned above, namely, the late-time emergence of cosmological gravitational radiation background. The cosmological classification of gravitational fields will be given in terms of time rather than in terms of the radial distance as one usually has in general relativity (Petrov, 1969). This is done for the inhomogeneous generalizations of Bianchi types I–VII. An immediate conclusion of our analysis is that the dominant type of gravitational field, which survives as time goes to infinity, is the radiative field.

In Section 2 we discuss the so-called generalized Einstein–Rosen metrics. In Section 3 we review the asymptotic properties of gravitational fields in classical general relativity theory. In Section 4 the cosmological peeling-off behavior is formulated, whereas Section 5 is devoted to the concluding remarks.

2. GENERALIZED EINSTEIN–ROSEN METRICS

In this section a brief outline of the so-called generalized Einstein–Rosen metrics is given. For more details on the subject the reader is referred to the review article by Carmeli and Charach (1984).

Owing to the mathematical complexity of the inhomogeneous cosmological models, one usually confines oneself to space-times which are nonuniform in one spatial direction and uniform in the other two directions. Such space-times are described by the generalized Einstein–Rosen metrics, given by the line element,

$$\frac{ds^2}{L^2} = e^{2f}(dz^2 - dt^2) + \gamma_{ab}\, dx^a\, dx^b \tag{1}$$

Here L is a unit length, $t = x^0$ is a timelike coordinate, $a, b = 2, 3$, the signature is $+2$, and f and γ_{ab} are functions of the coordinates z and t only.

The metric (1) admits two commuting Killing vectors, $Y^{\mu}_{(1)} = \delta^{\mu}_{2}$ and $Y^{\mu}_{(2)} = \delta^{\mu}_{3}$. The element of the transitivity surface G is given by

$$G = [(Y_{(1)\mu} Y_{(2)\nu} - Y_{(1)\nu} Y_{(2)\mu}) Y^{\mu}_{(1)} Y^{\nu}_{(2)}]^{1/2} = \gamma^{1/2} \tag{2}$$

where

$$\gamma = \det \gamma_{ab} \tag{3}$$

The local behavior of the models described by the line element (1) is determined by the four-vector $G_\mu = G_{,\mu}$ which can be timelike, spacelike, or null. Globally spacelike and null G_μ correspond to solutions describing cylindrical and plane gravitational waves. In cosmology, however, one is interested in a globally timelike vector G_μ, or in the more general case for which G_μ may vary from point to point.

Owing to the presence of the Abelian subgroup G_2 in the Bianchi models of types I–VII and in the axially symmetric Bianchi types VIII and IX, the Einstein–Rosen metrics include these models and some of their inhomogeneous generalizations as particular cases. This fact, which was first pointed out by Tomita (1978), makes them very relevant to the study of the inhomogeneous cosmologies.

The Einstein field equations in vacuum have the form (Carmeli et al., 1981)

$$(\sqrt{\gamma}\kappa_a^b)^{\cdot} - (\sqrt{\gamma}\lambda_a^b)' = 0 \tag{4}$$

$$-\frac{\kappa}{4}f' - \frac{\lambda}{4}\dot{f} + \kappa' + \tfrac{1}{2}\kappa_b^a\lambda_a^b = 0 \tag{5}$$

$$-\frac{\lambda}{2}f' - \frac{\kappa}{2}\dot{f} + \dot{\kappa} + \lambda' + \tfrac{1}{2}\kappa_b^a\kappa_a^b + \tfrac{1}{2}\lambda_a^b\lambda_b^a = 0 \tag{6}$$

$$(\ddot{f} - f'') + \dot{\kappa} + \tfrac{1}{2}\kappa_b^a\kappa_a^b - \tfrac{1}{2}\lambda_a^b\lambda_b^a = 0 \tag{7}$$

where

$$\begin{aligned} \kappa_{ab} &= \dot{\gamma}_{ab}, & \lambda_{ab} &= \gamma'_{ab} \\ \kappa &= \gamma^{ab}\kappa_{ab}, & \lambda &= \gamma^{ab}\lambda_{ab} \end{aligned} \tag{8}$$

Here dots and primes denote derivatives with respect to t and z.

It can be shown, without lost of generality (Carr and Verdaguer, 1983), that Bianchi cosmological models of types I–VII, as well as their unidirectional inhomogeneous generalizations, correspond to a special choice of γ_{ab} satisfying

$$\det \gamma_{ab} = t^2 \tag{9}$$

Likewise, choosing $\det\gamma_{ab}$ to be proportional to $\sin^2 t$ and $\sinh^2 t$ correspond, respectively, to axially symmetric Bianchi type IX and VIII cosmological models and their generalizations (Carmeli and Charach, 1984; Carmeli et al., 1983a, b).

Of particular interest is the metric

$$ds^2 = e^{2f}(dz^2 - dt^2) + e^{2\psi}\,dx^2 + 4t\chi\,dx\,dy + t^2e^{-2\psi}(1+4\chi^2)\,dy^2 \tag{10}$$

Here, f, ψ, and χ are functions of z and t only. The line element (10) is easily recognized as one of the forms of the generalized Einstein–Rosen metric with $\det\gamma_{ab} = t^2$, thus including Bianchi models of types I–VII along with their generalizations. Both of the gravitational degrees of freedom χ and ψ are chosen in such a way that they have the same asymptotic form when $t \to \infty$. The choice of $\det\gamma_{ab} \simeq \sin^2 t$ (which corresponds to close universes) is unsuitable for discussing gravitational radiation using asymptotic expansions in terms of time, since in this case the Universe does not expand to infinity. Neither will be considered the case for which $\det\gamma_{ab}$ corresponds to generalized Bianchi type-VIII models, because of extreme mathematical difficulties.

3. THE ASYMPTOTIC PROPERTIES OF GRAVITATIONAL FIELDS

The peeling-off behavior of the Riemann tensor in general relativity theory was first pointed out by Sachs (1962) in connection with the study of the asymptotic properties of gravitational radiation fields. It is convenient to expand the physical quantities, associated with the field, in powers of $1/r$, where r is the affine distance from an isolated system of sources. The calculations are conveniently performed in the framework of the so-called Newman–Penrose formalism (Newman and Penrose, 1962), where one studies the tetrad components of the Riemann tensor instead of its coordinate components thus avoiding the choice of a "preferable" coordinate system. In this method a tetrad (l^μ, n^μ, m, $\bar{m}^\mu$) of null vectors is introduced, satisfying

$$\begin{aligned} &l^\mu m_\mu = n^\mu m_\mu = 0 \\ &l^\mu n_\mu = 1, \qquad m^\mu \bar{m}_\mu = -1 \end{aligned} \tag{11}$$

where $\bar{m}^\mu$ is the complex conjugate of m^μ, and l^μ and n^μ are real. Then the metric $g^{\mu\nu}$ can be expressed as

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu \tag{12}$$

The ten real components of the Riemann tensor in vacuum are now specified uniquely by five complex scalars:

$$\begin{aligned} \Psi_0 &= -R_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma \\ \Psi_1 &= -R_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma \\ \Psi_2 &= -\tfrac{1}{2} R_{\mu\nu\rho\sigma} l^\mu n^\nu (l^\rho n^\sigma - m^\rho \bar{m}^\sigma) \\ \Psi_3 &= -R_{\mu\nu\rho\sigma} \bar{m}^\mu n^\nu l^\rho n^\sigma \\ \Psi_4 &= -R_{\mu\nu\rho\sigma} \bar{m}^\mu n^\nu \bar{m}^\rho n^\sigma \end{aligned} \tag{13}$$

It can be shown (Sachs, 1962; Newman and Penrose, 1962) that under certain general assumptions of approach to flatness at infinity, which is natural for radiative empty spaces, the Riemann tensor exhibits a characteristic asymptotic behavior of the form

$$\Psi_n = O(r^{n-5}), \qquad n = 0, 1, \ldots, 4 \tag{14}$$

Combining the latter result with the Petrov algebraic classification of the gravitational fields (Petrov, 1969) then allows one to derive the following general decomposition of the Riemann tensor (Sachs, 1962):

$$R_{\mu\nu\rho\sigma} = N_{\mu\nu\rho\sigma} r^{-1} + \mathrm{III}_{\mu\nu\rho\sigma} r^{-2} + \mathrm{II}_{\mu\nu\rho\sigma} r^{-3} + I_{\mu\nu\rho\sigma} r^{-4} + I'_{\mu\nu\rho\sigma} r^{-5} \tag{15}$$

Here the coefficients $N_{\mu\nu\rho\sigma}, \ldots, I_{\mu\nu\rho\sigma}$ denote fields of types $N, \ldots, I$ according to the Petrov classification.

The physical significance of the decomposition (15) is clarified when one compares it to the analogous behavior of the Maxwell electromagnetic tensor $F_{\mu\nu}$ of the fields of an isolated charge-current distribution. One then obtains (Goldberg and Kerr, 1964)

$$F_{\mu\nu} = N_{\mu\nu} r^{-1} + \mathrm{III}_{\mu\nu} r^{-2} + O(r^{-3}) \tag{16}$$

where the coefficients satisfy the conditions along the null rays k^μ,

$$N_{\mu\nu} k^\nu = 0, \qquad \mathrm{III}_{\mu\nu} k^\nu = A k_\mu, \qquad k_\mu k^\mu = 0 \tag{17}$$

It is well known from classical electrodynamics that one interprets the Maxwell tensor in equation (16) in terms of the near zone (induction zone), and the far zone (wave zone) dominated by the radiative N-type electromagnetic field. This allows one to interpret the results of the decomposition (15) for the gravitational field in the same way. Very far from the gravitationally emitting system at distances much greater than the dimensions of the system itself and the emitted wavelength, the local stationary observer will "experience" approximately the N radiative type field. Near the emitting system at distances smaller than the system dimensions and the emitted wavelength, the terms of the type-I (algebraically general) field will dominate. At distances smaller than the wavelength of emission but greater than the dimensions of the system, the terms of types II and III will be most important.

One may extend these results to the nonempty spaces by including matter. The behavior of gravitational fields of electrically charged bodies is also determined by the Sachs peeling-off theorem (Kozarzewski, 1965), since trajectories of propagation of gravitational and electromagnetic radiation are the same. Hawking (1968) has investigated the peeling-off behavior of outgoing gravitational radiation from bounded sources in a dust-filled open Friedmann universe. He showed that the peeling-off behavior still holds in an expanding universe, although the decomposition in the powers of $1/r$ is changed by the presence of matter.

The asymptotic behavior of the Riemann tensor in the cylindrical case was analyzed in detail by Stachel (1966), and the Riemann tensor was found to obey the decomposition in powers of $1/r^{1/2}$ rather than in powers of $1/r$.

As was pointed out in Section 2, the generalized Einstein–Rosen metrics describe equally well the cylindrical space-times and the cosmological models based on all Bianchi types. Consequently, the results obtained by us (Carmeli and Feinstein, 1984) are bound to be similar to those obtained by Stachel (1966), though in a very different physical context.

4. THE COSMOLOGICAL PEELING-OFF THEOREM

The physical behavior of the cosmological models described by the metric (10) may best be understood by examining the components of the Riemann tensor, taken with respect to the null tetrad l^μ, n^μ, m^μ, and $\bar{m}^\mu$ given by

$$\begin{gathered} l^\mu = e^{-f}(\delta^\mu_z + \delta^\mu_t), \qquad n^\mu = \tfrac{1}{2}e^{-f}(\delta^\mu_z - \delta^\mu_t) \\ m^\mu = 2^{-1/2}[e^{-\psi}\delta^\mu_x + i(\delta^\mu_y + 2te^{-2\psi}\chi\delta^\mu_x)e^{\psi}/t] \end{gathered} \tag{18}$$

with $x^\mu = (t, z, x, y)$. Analytic expressions for the complex components of the Riemann tensor Ψ_n, $n = 0, \ldots, 4$, were given by Stachel (1966), and they are rather lengthy.

It follows that for the entire class of metrics (10) the only nonzero components of the Riemann tensor are Ψ_0, Ψ_2, and Ψ_4. This easily enables one to find the Petrov type from the Ψ's. Since $\Psi_1 = \Psi_3 = 0$, one can have only four distinct, or two double, roots to the quartic algebraic equation (Kramer et al., 1980)

$$\Psi_0 + 6\Psi_2 E^2 + \Psi_4 E^4 = 0 \tag{19}$$

Thus the metric should be of type I or D. For the homogeneous axisymmetric Kasner solution ($\psi = \frac{1}{2}\ln t$, $\chi = 0$) and for the flat space-time ($\psi = 0$, $\chi = 0$ or $\psi = \ln t$, $\chi = 0$) the metric is algebraically special. All other solutions examined have proved to be of general Petrov type I.

As was pointed out in Section 3, the essential of the peeling-off behavior is that various different components of the Riemann tensor behave in different powers of the affine distance. In cosmology, however, one is interested in the development of the model with time rather than with the spatial distance.

To study such a behavior of the Riemann tensor we therefore expand the Ψ's in a power series in $t^{-1/2}$ rather than in $\rho^{-1/2}$ for Stachel's cylindrical case (1966). We also notice that both gravitational degrees of freedom $\psi_{\rm rad}$ and $\chi_{\rm rad}$ describing radiation behave like $O(t^{-1/2})$ as time goes to infinity. For the nonvanishing components of the Riemann tensor Ψ_0, Ψ_2, and Ψ_4 we find, as $t\to\infty$,

$$\begin{aligned}\Psi_0 &= e^{2f}[O(1/t^{5/2})+a(a-1)(2a-1)/t^2]\\ \Psi_2 &= e^{2f}O(1/t^{3/2}), \qquad \Psi_4 = e^{2f}O(1/t^{1/2})\end{aligned} \tag{20}$$

where a is related to the homogeneous mode of the gravitational field, $\psi = a \ln t$, similarly to Stachel's cylindrical case but with the time coordinate t replacing the spatial coordinate ρ.

Equations (20) show that the properties of the radiation field are strongly influenced by the presence of the homogeneous mode which gives rise to the term $a(a-1)(2a-1)/t^2$ and slows down the fall-off of many physical quantities as $t\to\infty$. There are, however, some exceptions when $a=0, 1, \frac{1}{2}$, which correspond to the flat ($a=0, 1$) and axisymmetric Kasner ($a=\frac{1}{2}$) spacetimes. For the homogeneous solutions, the factor e^{2f} attends its maximum value $t^{1/2}$ in the axisymmetric Kasner case ($a=\frac{1}{2}$), which means that the Riemann tensor at $t\to\infty$ always vanishes except in this case. At any rate, to the orders $e^{2f}/t^{1/2}$ and e^{2f}/t, the only nonvanishing component of Ψ_n is Ψ_4 and therefore the metric is of Petrov type N. To order $e^{2f}/t^{3/2}$, Ψ_2 and Ψ_4 do not vanish and the metric is of type II. To order e^{2f}/t^2, the metric may still be of type II, depending on whether or not the coefficient $a(a-1)(2a-1)$ in Ψ_0 vanishes. As has been previously pointed out, the vanishing of the coefficient corresponds either to the axisymmetric Kasner or flat space-times. In all the other cases the metric is of general Petrov type I.

To summarize the above we can write, in complete analogy to Stachel's cylindrical case,

$$\begin{aligned}e^{-2f}R_{\alpha\beta\gamma\delta} &= {}_0N_{\alpha\beta\gamma\delta}/t^{1/2}+{}_0N'_{\alpha\beta\gamma\delta}/t+{}_0\mathrm{II}_{\alpha\beta\gamma\delta}/t^{3/2}\\ &\quad+{}_0I(\mathrm{II}, D)_{\alpha\beta\gamma\delta}/t^2+{}_0I_{\alpha\beta\gamma\delta}/t^{5/2}\end{aligned} \tag{21}$$

The space-time may thus be divided into three different zones as time approaches infinity, in the last one of which the field is pure radiative.

5. CONCLUDING REMARKS

We have shown explicitly that the Riemann tensor of a large class of inhomogeneous cosmologies has a peeling-off behavior in time. This was proved for unidirectional generalizations of Bianchi types I–VII cosmological models. Our results are bound to be mathematically similar to those obtained by Stachel (1966) for the standard Einstein–Rosen cylindrical case. This is so since the metric used here is effectively the same as Stachel's, with the two nonignorable coordinates being swapped around. The physical interpretation of the Einstein–Rosen metric as a cosmological model developing in time, however, is preferable as compared to the standard cylindrical case which is physically unrealistic.

Our results allow one to classify cosmological gravitational fields in time, and to establish, in some sense, the algebraic history for a quite general class of inhomogeneous universes. Near the Big Bang singularity, as $t \to 0$, these universes behave chaotically, whereas in the late-time limit one finds pure gravitational radiation.

Although the peeling-off behavior is proved for the vacuum universes, one may easily extend it to the case of electromagnetic cosmologies [along the lines of the work (Kozarzewski, 1965)], and also to the case of universes filled with stiff fluid. In the latter case the presence of the matter will only slightly change the scale factor e^{2f} (Carmeli et al., 1981, 1983a, b), thus producing no influence on the peeling-off behavior of the Riemann tensor.

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